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Nonlinear Observer Design for a General Class of Nonlinear Systems with Real Parametric Uncertainty

V. SUNDARAPANDIAN
Department of Instrumentation and Control Engineering
SRM Institute of Science and Technology
SRM Nagar, Kattankulathur-603 203
Tamil Nadu, India
sundarsrm@yahoo.co.in

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Abstract—This paper is a geometric study of the local observer design for a general class of nonlinear systems with real parametric uncertainty. Explicitly, we study the observer design problem for a general class of nonlinear systems with real parametric uncertainty and with an input generator (exosystem). In this paper, we show that for the classical case, when the state equilibrium does not change with the parametric uncertainty, and when the plant output is purely a function of the state, there is no local asymptotic observer for the plant. Next, we show that in sharp contrast to this case, for the general case of problems where we allow the state equilibrium to change with the parametric uncertainty, there typically exist local exponential observers even when the plant output is purely a function of the state. We also present a characterization and construction procedure for local exponential observers for the general class of nonlinear systems with real parametric uncertainty under some stability assumptions. We also show that for the general class of nonlinear systems considered, under some stability assumptions, the existence of local exponential observers in the presence of inputs implies, and is implied by, the existence of local exponential observers in the absence of inputs. Finally, we generalize our results to a general class of nonlinear systems with input generator, and with exogenous disturbance.

Keywords—Exponential observers, Asymptotic observers, Real parametric uncertainty, Detectability, Exogenous disturbance.

1. INTRODUCTION

The nonlinear observer design problem was introduced by Thau [1]. Over the past three decades, there has been a significant attention paid in the control literature to the construction of observers for nonlinear systems [2–13].

This paper is a geometric study of the nonlinear observer design problem for a general class of nonlinear systems with real parametric uncertainty. In this paper, we extend our recent results on the nonlinear observer design problem [14–16]. Explicitly, the general class considered is the

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class of nonlinear systems of the form

$$\begin{aligned} \dot{x} &= f(x, \lambda) + g(x, \lambda)u, \\ y &= h(x, \lambda), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the *state*, $\lambda \in \mathbb{R}^l$ the *real parametric uncertainty*, $u \in \mathbb{R}^m$ the *input*, and $y \in \mathbb{R}^p$ the *output*. We assume that the state x is defined in an open neighborhood X of the origin of $\mathbb{R}^n$, and the input $u(\cdot)$ belongs to a class $\mathcal{U}$ of admissible input functions. In Sections 2 and 3, we assume that $\mathcal{U}$ consists of all locally C^1 functions $u(\cdot)$ with $u(0) = 0$. In Section 4, we assume that $\mathcal{U}$ consists of all inputs of the form

$$u = r(\omega), \tag{2}$$

where $\omega \in \mathbb{R}^q$ is the state of an *input generator* (exosystem) given by the dynamics

$$\dot{\omega} = s(\omega). \tag{3}$$

In Section 4, we will assume that the dynamics (3) is *neutrally stable* at $\omega = 0$. If the dynamics (3) is neutrally stable at $\omega = 0$, then it follows that it is Lyapunov stable at $\omega = 0$, and also that the linearization matrix $S = \frac{\partial s}{\partial \omega}(0)$ has all eigenvalues on the imaginary axis.

We also assume that the parametric uncertainty λ takes values in an open neighborhood Θ of the origin of $\mathbb{R}^l$. We set $Y \triangleq h(X, \Theta)$. We also assume that

$$f(0,0) = 0, \qquad g(0,0) = 0, \qquad \text{and} \qquad h(0,0) = 0.$$

In our recent paper [15] on the observer design problem, we derived various results for the unforced case, i.e., when $\mathcal{U} = \{0\}$. In this paper, we extend the results in [15] for the class of nonlinear systems with inputs (1).

In this paper, we first show that *zero-state detectability* is a necessary condition for the existence of local asymptotic observers for the nonlinear system (1). Using this necessary condition, we establish that for the classical case of problems, when the state equilibrium does not change with the real parametric uncertainty, there does not exist any local asymptotic observer for the nonlinear plant. Next, we show that in sharp contrast to this case, for the general case of problems where we allow the state equilibrium to change with the real parametric uncertainty, there typically exist local exponential observers even when the plant output is purely a function of the state.

Under some stability assumptions on the plant, we also derive a result giving necessary and sufficient conditions for local exponential observers for the given nonlinear plant with exogenous inputs. Using this characterizing result, we also derive a simple construction procedure for designing exponential observers for the given nonlinear plant with exogenous inputs. In this context, we also derive an interesting result which states that under some stability assumptions on the plant, the existence of local exponential observers for the nonlinear plant (1) in the presence of inputs implies, and is implied by, the existence of local exponential observers for plant (1) in the absence of inputs. Thus, this result simplifies the nonlinear observer design problem significantly. We illustrate our various results with examples.

This paper is organized as follows. In Section 2, we define local asymptotic and exponential observers for the general class of nonlinear systems with real parametric uncertainty. In Section 3, we derive a simple necessary condition, namely zero-state detectability, for local asymptotic observers for the general class of nonlinear systems with real parametric uncertainty, and discuss its consequences. In Section 4, we derive necessary and sufficient conditions for local exponential observers for the general class of nonlinear systems with real parametric uncertainty. We derive a set of important results on observer design problem, and illustrate them with examples. Finally, in Section 5, we extend our results to a general class of nonlinear systems with exogenous disturbance.

2. BASIC DEFINITIONS

In this paper, we are interested in the nonlinear observer design problem for the nonlinear plant (1). Since λ is a real parametric uncertainty, it is not known, and hence it is beneficiary to consider λ as an additional state variable. Thus, we consider plant (1) in the form

$$\begin{aligned} \dot{x} &= f(x,\lambda) + g(x,\lambda)u, \\ \dot{\lambda} &= 0, \\ y &= h(x,\lambda). \end{aligned} \tag{4}$$

In this paper, we derive results for local asymptotic observers and local exponential observers for the nonlinear plant (4) with real parametric uncertainty. Local asymptotic and exponential observers are defined as in [2,7,14–16].

DEFINITION 1. *Consider the nonlinear system (candidate observer) described by*

$$\begin{aligned} \dot{z} &= \phi(z,\mu,y,u), \\ \dot{\mu} &= \psi(z,\mu,y,u), \end{aligned} \tag{5}$$

where the state z of the candidate observer (5) is defined locally (say, in the neighborhood X) of the origin of $\mathbb{R}^n$, and the state μ of the candidate observer (5) is defined locally (say, in the neighborhood Θ) of the origin of $\mathbb{R}^l$. We assume that ϕ and ψ are locally C^1 mappings, such that $\phi(0,0,0,0) = 0$ and $\psi(0,0,0,0) = 0$. We say that the candidate observer (5) is a local asymptotic (respectively, exponential) observer for plant (4), if the composite system (4),(5) satisfies the following two conditions.

(O1) *If*

$$\begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix} = \begin{bmatrix} z(0) \\ \mu(0) \end{bmatrix},$$

then

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} z(t) \\ \mu(t) \end{bmatrix},$$

for all $t \geq 0$ and for all $u(\cdot) \in \mathcal{U}$.

(O2) *There exists a neighborhood V of the origin of $\mathbb{R}^n \times \mathbb{R}^l$, such that for all values of*

$$\begin{bmatrix} z(0) \\ \mu(0) \end{bmatrix} - \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix}$$

in V, the measurement error

$$\left\| \begin{bmatrix} z(t) - x(t) \\ \mu(t) - \lambda(t) \end{bmatrix} \right\| \to 0$$

asymptotically (respectively, exponentially) as $t \to \infty$. ∎

REMARK 1. There are some important cases of interest included in Definition 1. First, the case $\mathcal{U} = \{0\}$ corresponds to the problem of finding local asymptotic and exponential observers for unforced dynamical systems, which was discussed in detail in our recent paper [15]. Other important special cases of interest are constant inputs, and periodic inputs, both of which are treated in Section 4. ∎

We define the *estimation error* e by

$$e \triangleq \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} z \\ \mu \end{bmatrix} - \begin{bmatrix} x \\ \lambda \end{bmatrix}.$$

Then the error e satisfies the dynamics

$$\begin{aligned}
\dot{e}_1 &= \phi(x + e_1, \lambda + e_2, y, u) - f(x, \lambda) - g(x, \lambda)u, \\
\dot{e}_2 &= \psi(x + e_1, \lambda + e_2, y, u).
\end{aligned}$$

We consider the composite system

$$\begin{aligned}
\dot{x} &= f(x, \lambda) + g(x, \lambda)u, \\
\dot{\lambda} &= 0, \\
\dot{e}_1 &= \phi(x + e_1, \lambda + e_2, h(x, \lambda), u) - f(x, \lambda) - g(x, \lambda)u, \\
\dot{e}_2 &= \psi(x + e_1, \lambda + e_2, h(x, \lambda), u).
\end{aligned} \tag{6}$$

Next, we state a simple lemma which provides a geometric characterization of Condition (O1) in Definition 1.

LEMMA 1. *(See [16].) The following statements are equivalent.*

(a) *Condition (O1) in Definition 1 holds for the composite system (4),(5).*

(b) $\phi(x, h(x, \lambda), u) = f(x, \lambda) + g(x, \lambda)u$ *and* $\psi(x, h(x, \lambda), u) \equiv 0$ *for all* $x \in X$, $\lambda \in \Theta$ *and for all* $u(\cdot) \in \mathcal{U}$.

(c) *The submanifold defined via* $e = 0$ *is invariant under the flow of the composite system (6).* ∎

As a simple consequence of Lemma 1, we have the following result.

LEMMA 2. *(See [7, Theorem 1].) Consider plant (4) and the candidate observer (5). Then Condition (O1) holds if and only if* ϕ *and* ψ *have the following form:*

$$\begin{aligned}
\phi(z, \mu, y, u) &= f(z, \mu) + g(z, \mu)u + \alpha(z, \mu, y, u), \\
\psi(z, \mu, y, u) &= \beta(z, \mu, y, u),
\end{aligned}$$

where α *and* β *are locally* C^1 *mappings with* $\alpha(0,0,0,0) = 0$, $\beta(0,0,0,0) = 0$, *and also such that*

$$\begin{aligned}
\alpha(x, \lambda, h(x, \lambda), u) &= 0, \\
\beta(x, \lambda, h(x, \lambda), u) &= 0.
\end{aligned}$$ ∎

3. A NECESSARY CONDITION FOR ASYMPTOTIC OBSERVERS FOR GENERAL NONLINEAR SYSTEMS WITH REAL PARAMETRIC UNCERTAINTY

In this section, we shall show that *zero-state detectability* is a necessary condition for the existence of a local asymptotic observer for the nonlinear plant (4). Using this necessary condition, we shall establish that in the classical case where the state equilibrium does not change with the real parametric uncertainty, and the plant output is purely a function of the state x, i.e., y is of the form $y = \gamma(x)$, there is no local asymptotic observer for the plant. In Section 4, we shall show that in sharp contrast to this case, for the general case, where we allow the state equilibrium to change with the real parametric uncertainty, there typically exist local exponential observers even when the plant output y has the form $y = \gamma(x)$.

In the next result, we will show that if plant (4) has a local asymptotic observer of form (5), then plant (4) is *zero-state detectable*, i.e., for any solution $(x(t), \lambda(t))$ of (4) with small initial condition $(x(0), \lambda(0)) = (x_0, \lambda_0)$, such that $y(t) = h(x(t), \lambda(t)) \equiv 0$, then we must have $(x(t), \lambda(t)) \to 0$ asymptotically as $t \to \infty$. Since $\lambda(t) \equiv \lambda_0$, the zero-state detectability requirement amounts to requiring that the solution $(x(t), \lambda(t))$ yielding zero-output for the plant (4) is such that $x(t) \to 0$ as $t \to \infty$ and $\lambda_0 = 0$.

THEOREM 1. *A necessary condition for the existence of a local asymptotic observer for the plant (4) is that plant (4) is zero-state detectable, i.e., any solution trajectory $(x(t), \lambda(t))$ of (4) with small initial condition (x_0, λ_0) satisfying*

$$y(t) = h(x(t), \lambda(t)) \equiv 0$$

must be such that $x(t) \to 0$ asymptotically as $t \to \infty$ and $\lambda_0 = 0$.

PROOF. Suppose that the candidate observer defined by (5) forms a local asymptotic observer for plant (4). Then Conditions (O1) and (O2) are satisfied. Let $(x(t), \lambda(t))$ be any state trajectory of (4) with small initial condition (x_0, λ_0) satisfying

$$y(t) = h(x(t), \lambda(t)) \equiv 0.$$

Then by Lemma 2, the observer dynamics in (5) takes the form

$$\begin{aligned} \dot{z} &= \phi(z, \mu, 0, u) = f(z, \mu) + g(z, \mu)u + \alpha(z, \mu, 0, u), \\ \dot{\mu} &= \psi(z, \mu, 0, u) = \beta(z, \mu, 0, u). \end{aligned} \tag{7}$$

Taking $z_0 = 0$ and $\mu_0 = 0$, it follows that the observer trajectory $(z(t), \mu(t))$ satisfies $z(t) \equiv 0$ and $\mu(t) \equiv 0$. Hence, by Condition (O2) of Definition 1 for local asymptotic observers, it follows that

$$\|(x(t), \lambda(t))\| \to 0, \qquad \text{as } t \to \infty.$$

Since $\|x(t)\| \leq \|(x(t), \lambda(t))\|$, it also follows that $x(t) \to 0$ asymptotically as $t \to \infty$. Since $\lambda(t) \equiv \lambda_0$, it follows that we must have $\lambda_0 = 0$. This completes the proof. ∎

Using Theorem 1, we can prove the following result which says that there is no local asymptotic observer for plant (4) if the equilibrium $x = 0$ does not change with the real parametric uncertainty, i.e., $f(0, \lambda) \equiv 0$ and $g(0, \lambda) \equiv 0$, and if the output function y is purely a function of x, i.e., $y = h(x, \lambda) = \gamma(x)$.

Before we prove this result, we observe that there are many plants in nonlinear control systems which satisfy the assumption

$$f(0, \lambda) \equiv 0 \qquad \text{and} \qquad g(0, \lambda) \equiv 0. \tag{8}$$

In classical bifurcation theory, a standard assumption is that there is a trivial solution from which the bifurcation is to occur [17, p. 149]. Thus, in the classical bifurcation case, the control plant (4) is often assumed to satisfy (8) for all λ so that $x = 0$ is an equilibrium for all parameter values.

Another class of plants which satisfy the assumption (8) is one in which

$$f(x, \lambda) = f_a(x) + f_b(x)\lambda \qquad \text{and} \qquad g(x, \lambda) = g_a(x) + g_b(x)\lambda,$$

with $f_a(0) = 0$, $f_b(0) = 0$, $g_a(0) = 0$, and $g_b(0) = 0$.

THEOREM 2. *Suppose that plant (4) satisfies assumption (8) so that $x = 0$ is an equilibrium for all values of the parameter λ and also that the output function y is purely a function of x, i.e., it has the form $y = \gamma(x)$. Then there is no local asymptotic observer for plant (4).*

PROOF. This is an immediate consequence of Theorem 1. We show that plant (4) is not zero-state detectable. This is easily seen by taking $x(0) = x_0 = 0$ and $\lambda(0) = \lambda_0 \neq 0$ for any λ_0 arbitrarily small. Then we have $x(t) \equiv 0$ for all t, and so it follows that

$$y(t) = h(x(t), \lambda(t)) = \gamma(x(t)) \equiv 0.$$

However, $\lambda(t) \equiv \lambda_0 \neq 0$. This shows that plant (4) is not zero-state detectable. From the necessary condition given in Theorem 1, it follows that there is no local asymptotic observer for plant (4). This completes the proof. ∎

COROLLARY 1. *Under the same assumptions in Theorem 2, there is no local exponential observer for plant (4).* ∎

Theorem 2 and Corollary 1 are for the classical case of nonlinear control systems with real parametric uncertainty, i.e., systems of form (4) satisfying assumption (8). In the next section, we shall show that in sharp contrast to this case, for the general case of nonlinear control systems with real parametric uncertainty, where we allow the state equilibrium to change with the parameter, there typically exist local exponential observers even when the output function y is purely a function of x.

4. A CHARACTERIZATION AND CONSTRUCTION OF EXPONENTIAL OBSERVERS FOR GENERAL NONLINEAR SYSTEMS WITH REAL PARAMETRIC UNCERTAINTY

In this section, we suppose that the class $\mathcal{U}$ consists of inputs $u(\cdot)$ of the form

$$u = r(\omega), \tag{9}$$

where ω satisfies the autonomous system (*exosystem*)

$$\dot{\omega} = s(\omega), \qquad \text{with } s(0) = 0. \tag{10}$$

The state ω of the exosystem lies in an open neighborhood W of the origin in $\mathbb{R}^q$. One can view equations (9) and (10) as an *input generator*. We assume that the exosystem dynamics (10) is *neutrally stable* at $\omega = 0$. Basically, it means that the dynamics (10) is Lyapunov stable in both forward and backward time at $\omega = 0$. As a consequence of this definition, it follows that the linearization of the exosystem dynamics $\dot{\omega} = S\omega$ is such that S has all eigenvalues on the imaginary axis.

First, we present a basic theorem that completely characterizes the existence of local exponential observers of form (5) for nonlinear plants of form (4). We note that this result holds for both classical and general cases of problems.

Using (9) and (10), plant (4) can be expressed as

$$\begin{aligned} \dot{x} &= f(x, \lambda) + g(x, \lambda) r(\omega), \\ \dot{\lambda} &= 0, \\ \dot{\omega} &= s(\omega), \\ y &= h(x, \lambda). \end{aligned} \tag{11}$$

Also, the composite system (6) can be expressed as

$$\begin{aligned} \dot{x} &= f(x, \lambda) + g(x, \lambda) r(\omega), \\ \dot{\lambda} &= 0, \\ \dot{\omega} &= s(\omega), \\ \dot{e}_1 &= \phi(x + e_1, \lambda + e_2, h(x, \lambda), r(\omega)) - f(x, \lambda) - g(x, \lambda) r(\omega), \\ \dot{e}_2 &= \psi(x + e_1, \lambda + e_2, h(x, \lambda), r(\omega)). \end{aligned} \tag{12}$$

We note that the estimation errors e_1 and e_2 are defined by $e_1 = z - x$ and $e_2 = \mu - \lambda$, respectively. There is no estimation required for ω, since ω is available for measurement.

THEOREM 3. *Suppose that the plant dynamics in (11) is Lyapunov stable at $(x, \lambda, \omega) = (0, 0, 0)$. Then the candidate observer (5) is a local exponential observer for plant (11) if, and only if,*

(a) *the submanifold defined via $e = 0$ is invariant under the flow of the composite system (12);*
(b) *the dynamics*

$$\begin{aligned} \dot{e}_1 &= \phi(e_1, e_2, 0, 0), \\ \dot{e}_2 &= \psi(e_1, e_2, 0, 0) \end{aligned} \tag{13}$$

is locally exponentially stable at $e = 0$.

PROOF. The necessity follows immediately from Definition 1 for local exponential observers, and Lemma 1. The sufficiency can be established using the center manifold theory [16] or using the Lyapunov stability theory [14, Theorem 6]. The proof is omitted, since it is very similar to the arguments given in our recent work [14, Theorem 6]. ∎

REMARK 2. The *Lyapunov stability* assumption on the dynamics (11) is not demanding too much. We note that our nonlinear observer design is carried out in a neighborhood of the origin. There is a conceptual problem in observer design, *viz.* what does the existence of a local exponential observer mean in terms of the nonlinear dynamics to be observed? For example, it must mean that the state trajectories do not have finite escape time, but what does local existence mean for unbounded trajectories? For this reason, we have focused our efforts in treating the local observer design problem on those nonlinear systems which are Lyapunov stable. This focus leads to a clearly posed observer design problem for which necessary and sufficient conditions can be derived. ∎

EXAMPLE 1. In the dynamics (11), if we set $\omega = 0$, then we get the unforced dynamics

$$\begin{aligned} \dot{x} &= f(x, \lambda), \\ \dot{\lambda} &= 0. \end{aligned} \tag{14}$$

Suppose that the zero-parameter unforced dynamics given by the equation

$$\dot{x} = f(x, 0) \tag{15}$$

is locally asymptotically stable at $x = 0$.

Next, suppose that the class $\mathcal{U}$ consists of inputs u of the form $u = r(\omega)$, where

$$\begin{aligned} \dot{\omega} &= S\omega, \\ S &= \begin{bmatrix} 0 & -\nu \\ \nu & 0 \end{bmatrix}, \\ \dot{\nu} &= 0. \end{aligned}$$

Clearly, $\mathcal{U}$ consists of periodic inputs with any desired period. Thus, it follows by a total stability result [18, p. 446, Corollary] that the plant dynamics in (11) is Lyapunov stable at $(x, \lambda, \omega) = (0, 0, 0)$ (by its triangular structure). Hence, the stability hypothesis made in Theorem 3 holds for this example. ∎

As an application of Theorem 3, we establish the following result which states that when the plant dynamics in (11) is Lyapunov stable at $(x, \lambda, \omega) = (0, 0, 0)$, the existence of a local exponential observer for plant (11) in the presence of inputs implies and is implied by the existence of a local exponential observer for plant (11) in the absence of inputs.

For the purpose of stating this result, we note that the unforced plant corresponding to $\omega = 0$ is given by

$$\begin{aligned} \dot{x} &= f(x, \lambda), \\ \dot{\lambda} &= 0, \\ y &= h(x, \lambda). \end{aligned} \tag{16}$$

THEOREM 4. *Suppose that the plant dynamics in (11) is Lyapunov stable at $(x, \lambda, \omega) = (0, 0, 0)$. If the system*

$$\begin{aligned} \dot{z} &= \phi(z, \mu, y, u), \\ \dot{\mu} &= \psi(z, \mu, y, u) \end{aligned}$$

is a local exponential observer for the full plant (11), then the system defined by

$$\dot{z} = \phi(z, \mu, y, 0),$$
$$\dot{\mu} = \psi(z, \mu, y, 0)$$

is a local exponential observer for the unforced plant (16). Conversely, if the system

$$\dot{z} = \eta(z, \mu, y),$$
$$\dot{\mu} = \sigma(z, \mu, y)$$

is a local exponential observer for the unforced plant (16), then the system defined by

$$\begin{bmatrix} \dot{z} \\ \dot{\mu} \end{bmatrix} = \begin{bmatrix} \phi(z, \mu, y, u) \\ \psi(z, \mu, y, u) \end{bmatrix} \triangleq \begin{bmatrix} \eta(z, \mu, y) \\ \sigma(z, \mu, y) \end{bmatrix} + \begin{bmatrix} g(z, y)u \\ 0 \end{bmatrix}$$

is a local exponential observer for the full plant (11).

PROOF. The first part of this theorem is straightforward. The second part of the theorem follows by verifying Conditions (a) and (b) given in Theorem 3. The calculations are omitted since they are very analogous to the calculations given in our recent work [14, Theorem 7]. ∎

REMARK 3. Theorem 4 is extremely useful in applications; it is a novel result in the nonlinear observer design problem for systems with inputs. For the class of nonlinear plants and exogenous inputs considered in this paper, Theorem 4 asserts that under the Lyapunov stability assumption on the plant dynamics (which is quite natural to ask for as pointed out in Remark 2), the existence of a local exponential observer for the full plant (11) in presence of inputs implies and is implied by the existence of local exponential observer for the full plant (11) in the absence of inputs. Theorem 4 also gives a simple construction procedure on constructing the local exponential observer for the full plant (11) from the local exponential observer for the unforced plant (16). The importance of this calculation is, of course, that the nonlinear observer design problem is considerably simplified, and we can derive simple a necessary and sufficient condition for exponential observers for the general class of nonlinear systems considered with real parametric uncertainty based on the system linearization matrices $C^\star$ and $A^\star$ for the unforced plant (16). In fact, we will show that under the Lyapunov stability assumption on the plant, a necessary and sufficient condition for the full plant (11) is that $(C^\star, A^\star)$ is detectable, i.e., there exists a matrix $K^\star$ for which $A^\star - K^\star C^\star$ is Hurwitz. If $(C^\star, A^\star)$ is observable, then it is well known that we can place the eigenvalues of $A^\star - K^\star C^\star$ arbitrarily in the complex plane, but subject to the conjugacy requirement. (The conjugacy requirement for eigenvalues is basically this fundamental property: if ξ is a complex eigenvalue of a square matrix, then its conjugate $\bar{\xi}$ is also an eigenvalue of the same matrix.) Thus, it can be easily seen that when $(C^\star, A^\star)$ is observable, we can construct a simple nonlinear observer with any desired speed of exponential decay of error. ∎

Let $(C^\star, A^\star)$ denote the linearization pair for the unforced plant (16), i.e.,

$$C^\star = \begin{bmatrix} C & Z \end{bmatrix} \qquad \text{and} \qquad A^\star = \begin{bmatrix} A & P \\ 0 & 0 \end{bmatrix},$$

where

$$C = \frac{\partial h}{\partial x}(0,0), \qquad Z = \frac{\partial h}{\partial \lambda}(0,0), \qquad A = \frac{\partial f}{\partial x}(0,0), \qquad \text{and} \qquad P = \frac{\partial f}{\partial \lambda}(0,0).$$

In view of the reduction procedure outlined in Theorem 4, we first state some important results on the exponential observer design for the unforced plant (16). First, we state the following necessary condition for local exponential observers that can be easily established as in [16].

THEOREM 5. *If the unforced plant (16) has a local exponential observer, then the pair* $(C^\star, A^\star)$ *is detectable.* ∎

COROLLARY 2. *If the full plant (11) has a local exponential observer, then the pair* $(C^\star, A^\star)$ *is detectable.*

PROOF. The assertion follows immediately from Theorems 4 and 5. ∎

Using the necessary condition given in Theorem 5, we can establish the following result, which gives a simple necessary condition for the existence of local exponential observers for the unforced plant (16).

THEOREM 6. *If the unforced plant (16) has a local exponential observer, then the pair* (C, A) *is detectable, and*

$$\operatorname{rank} \begin{bmatrix} Z \\ P \end{bmatrix} = l = \dim(\lambda).$$

PROOF. Suppose that the unforced plant (16) has a local exponential observer. Then by Theorem 5, the pair $(C^\star, A^\star)$ is detectable. Note that by the PBH rank test [19, p. 286], a necessary and sufficient condition for $(C^\star, A^\star)$ to be detectable is that

$$\operatorname{rank} \begin{bmatrix} C^\star \\ \xi I - A^\star \end{bmatrix} = n + l, \tag{17}$$

for all complex numbers ξ in the closed right half plane (RHP).

Since

$$\begin{bmatrix} C^\star \\ \xi I - A^\star \end{bmatrix} = \begin{bmatrix} C & Z \\ \xi I - A & -P \\ 0 & \xi I \end{bmatrix},$$

it is immediate that (32) holds for all complex numbers ξ in the closed RHP only if

$$\operatorname{rank} \begin{bmatrix} C \\ \xi I - A \end{bmatrix} = n,$$

for all complex numbers ξ in the closed RHP and

$$\operatorname{rank} \begin{bmatrix} Z \\ P \end{bmatrix} = l.$$

This completes the proof. ∎

COROLLARY 3. *If the full plant (11) has a local exponential observer, then the pair* (C, A) *is detectable, and*

$$\operatorname{rank} \begin{bmatrix} Z \\ P \end{bmatrix} = l = \dim(\lambda).$$

PROOF. This is a simple consequence of Theorems 4 and 6. ∎

Next, we show that the necessary condition given in Theorem 5 is also sufficient for the existence of a local exponential observer for the unforced plant (16) when the unforced plant dynamics in (16) is Lyapunov stable at $(x, \lambda) = (0, 0)$.

THEOREM 7. *Suppose that the plant dynamics in (16) is Lyapunov stable at* $(x, \lambda) = (0, 0)$, *and suppose also that for some* $(n + l) \times p$ *matrix* $K^\star$, *the matrix* $A^\star - K^\star C^\star$ *is Hurwitz. Then the system defined by*

$$\begin{bmatrix} \dot{z} \\ \dot{\mu} \end{bmatrix} = \begin{bmatrix} f(z, \mu) \\ 0 \end{bmatrix} + K^\star \left[y - h(z, \mu) \right] \tag{18}$$

is a local exponential observer for the unforced plant (16).

PROOF. It is easy to check that the candidate observer (18) satisfies Conditions (a) and (b) in Theorem 3. ∎

The next result makes use of the *reduction procedure* outlined in Theorem 4. When $(C^\star, A^\star)$ is detectable, by the reduction procedure outlined in Theorem 4, we can use the local exponential observer (18) constructed for the unforced plant (16) to construct a local exponential observer for the full plant (11).

THEOREM 8. *Suppose that the plant dynamics in (11) is Lyapunov stable at $(x, \lambda, \omega) = (0,0,0)$, and suppose also that for some $(n+l) \times p$ matrix $K^\star$, the matrix $A^\star - K^\star C^\star$ is Hurwitz. Then the system defined by*

$$\begin{bmatrix} \dot{z} \\ \dot{\mu} \end{bmatrix} = \begin{bmatrix} f(z,\mu) + g(z,\mu)u \\ 0 \end{bmatrix} + K^\star \left[y - h(z,\mu) \right] \tag{19}$$

is a local exponential observer for the full plant (11).

PROOF. The assertion is a simple consequence of the reduction procedure outlined in Theorems 4 and 7. ∎

COROLLARY 4. *Suppose that the plant dynamics in (11) is Lyapunov stable at $(x, \lambda, \omega) = (0,0)$, and that the output function y is purely a function of x, i.e., it has the form $y = \gamma(x)$. Assume that the equilibrium $x = 0$ of the plant dynamics of the full plant (11) changes with the real parametric uncertainty λ. In this case, the system pair $(C^\star, A^\star)$ has the form*

$$C^\star = \begin{bmatrix} C & 0 \end{bmatrix} \qquad \text{and} \qquad A^\star = \begin{bmatrix} A & P \\ 0 & 0 \end{bmatrix}.$$

If the pair $(C^\star, A^\star)$ is detectable, then the full plant (11) has a local exponential observer given by

$$\begin{bmatrix} \dot{z} \\ \dot{\mu} \end{bmatrix} = \begin{bmatrix} f(z,\mu) + g(z,\mu)u \\ 0 \end{bmatrix} + K^\star \left[y - \gamma(z) \right], \tag{20}$$

where $K^\star$ is any matrix, such that $A^\star - K^\star C^\star$ is Hurwitz. ∎

REMARK 4. It is well known that the system linearization pair $(C^\star, A^\star)$ is *generically* observable [20]. Thus, from Corollary 4, we deduce that under the conditions:

(a) the plant dynamics in (11) is Lyapunov stable at $(x, \lambda, \omega) = (0,0,0)$;
(b) the equilibrium $x = 0$ of the plant dynamics in (11) changes with the real parametric uncertainty;
(c) the output function y is purely a function of x, i.e., it has the form $y = \gamma(x)$,

there *generically* exist local exponential observers of form (20) for plant (11). ∎

Next, we show by an example that we can construct a local exponential observer for the full plant (11) when the stable equilibrium $x = 0$ changes with the real parametric uncertainty for the challenging case when the output function y is purely a function of the state x, i.e., giving no information on the real parametric uncertainty λ.

EXAMPLE 2. Consider a nonlinear system described by

$$\begin{aligned}
\dot{x}_1 &= x_2 + x_1 \left(\lambda - x_1^2 - x_2^2 \right) + \omega_1^2 x_1^2 - \omega_2^3 x_2 x_3, \\
\dot{x}_2 &= -x_1 + x_2 \left(\lambda - x_1^2 - x_2^2 \right) + \omega_1 \nu x_1 x_2 - \nu^2 x_3^2, \\
\dot{x}_3 &= -x_3 + \lambda + x_3^3 + \omega_1^3 \nu^2 x_2^4 - \omega_2^3 x_3^3 + \omega_1^3 \nu^2, \\
\dot{\lambda} &= 0, \\
\dot{\omega}_1 &= \nu \omega_2, \\
\dot{\omega}_2 &= -\nu \omega_1, \\
\dot{\nu} &= 0, \\
y_1 &= x_2 - x_1 x_2^2, \\
y_2 &= x_3 + x_2 x_3^2.
\end{aligned} \tag{21}$$

When we set $\omega_1 = \omega_2 = \nu = 0$ in (21), we get the *unforced plant* as

$$\begin{aligned}
\dot{x}_1 &= x_2 + x_1\left(\lambda - x_1^2 - x_2^2\right),\\
\dot{x}_2 &= -x_1 + x_2\left(\lambda - x_1^2 - x_2^2\right),\\
\dot{x}_3 &= -x_3 + \lambda + x_3^3,\\
\dot{\lambda} &= 0,\\
y_1 &= x_2 - x_1 x_2^2,\\
y_2 &= x_3 + x_2 x_3^2.
\end{aligned} \tag{22}$$

When $x = 0$, the vector field corresponding to x in (22) takes the form

$$f(0,\lambda) = \begin{bmatrix} 0 \\ 0 \\ \lambda \end{bmatrix}.$$

Thus, the equilibrium $x = 0$ of the state dynamics corresponding to x in (22) changes with the real parametric uncertainty λ.

It is clear that the dynamics in the unforced plant (22) exhibits a local codimension one bifurcation, namely a supercritical Hopf bifurcation [17, pp. 150–152]. We also note that the plant output has the form $y = \gamma(x)$. Nonetheless, we show that it is possible to construct a local exponential observer for the full plant (21).

In [15, Example 2], we constructed a local exponential observer for the unforced plant (22). Using the reduction procedure outlined in Theorem 4, we shall show that we can construct a local exponential observer for the full plant (21), given the local exponential observer for the unforced plant (22).

Similar to the calculations carried in [15, Example 2], we can easily show, by a total stability result [18, p. 515, Corollary], that the plant dynamics of the full plant (21) is Lyapunov stable at $(x, \lambda, \omega, \nu) = (0, 0, 0, 0)$.

Also, linearizing the unforced plant (22), we get the system matrices

$$C^\star = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad A^\star = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to check that $(C^\star, A^\star)$ is observable. In particular, $(C^\star, A^\star)$ is detectable. Indeed, setting

$$K^\star = \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \tag{23}$$

we see that the matrix $A^\star - K^\star C^\star$ is Hurwitz with eigenvalues -1, -1, -1, and -1.

Hence, by Corollary 4, it follows that a local exponential observer for the full plant (21) is given by the dynamics

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{\mu} \end{bmatrix} = \begin{bmatrix} z_2 + z_1\left(\mu - z_1^2 - z_2^2\right) + \omega_1^2 z_1^2 - \omega_2^3 z_2 z_3 \\ -z_1 + z_2\left(\mu - z_1^2 - z_2^2\right) + \omega_1 \nu z_1 z_2 - \nu^2 z_3^2 \\ -z_3 + \mu + z_3^3 + \omega_1^3 \nu^2 z_2^4 - \omega_2^3 z_3^3 + \omega_1^3 \nu^2 \\ 0 \end{bmatrix} + K^\star \begin{bmatrix} y_1 - z_2 + z_1 z_2^2 \\ y_2 - z_3 - z_2 z_3^2 \end{bmatrix},$$

where $K^\star$ is as defined by (23). ∎

5. NONLINEAR OBSERVERS FOR GENERAL NONLINEAR SYSTEMS WITH EXOGENOUS DISTURBANCE

In this section, we extend our results obtained in Sections 2–4 to nonlinear plants with exogenous disturbance, which have the general form

$$\begin{aligned} \dot{x} &= f(x,\lambda) + g(x,\lambda)u, \\ \dot{\lambda} &= \tau(\lambda), \\ y &= h(x,\lambda), \end{aligned} \tag{24}$$

where $x \in \mathbb{R}^n$ is the *state* of the plant, $u \in \mathbb{R}^m$ the *input*, $\lambda \in \mathbb{R}^l$ the *exogenous disturbance*, and $y \in \mathbb{R}^p$ the *output*. The exogenous variable λ satisfies the exosystem dynamics $\dot{\lambda} = \tau(\lambda)$, which is assumed to be *neutrally stable* at $\lambda = 0$, i.e., we assume that the exosystem dynamics $\dot{\lambda} = \tau(\lambda)$ is Lyapunov stable in both forward and backward time at $\lambda = 0$. The state x is defined in an open neighborhood X of the origin of $\mathbb{R}^n$, and the disturbance λ is defined in an open neighborhood Θ of the origin of $\mathbb{R}^l$. We assume that f, g, h, and τ are locally $\mathcal{C}^1$ mappings, and also that $f(0,0) = 0$, $g(0,0) = 0$, $\tau(0) = 0$, and $h(0,0) = 0$. We assume that $u(\cdot) \in \mathcal{U}$, the class of admissible inputs, where $\mathcal{U}$ consists of all locally $\mathcal{C}^1$ mappings $u(\cdot)$ with $u(0) = 0$. Since the disturbance λ is usually not known, it is beneficiary to treat λ as an additional state variable, and the observer we build for plant (24) will naturally seek estimates for both the state x and the disturbance λ.

We define local asymptotic and exponential observers for plant (24) by just replacing the dynamics $\dot{\lambda} = 0$ with $\dot{\lambda} = \tau(\lambda)$ in Definition 1 for local asymptotic and exponential observers. As in Section 2, we can easily prove the following necessary condition for plant (24) to have local asymptotic observers.

THEOREM 9. *Suppose that the exosystem dynamics $\dot{\lambda} = \tau(\lambda)$ is neutrally stable at $\lambda = 0$. A necessary condition for the existence of a local asymptotic observer for plant (24) is that plant (24) is zero-state detectable, i.e., any state trajectory $(x(t), \lambda(t))$ of (24) with small initial condition (x_0, λ_0) satisfying*

$$y(t) = h(x(t), \lambda(t)) \equiv 0$$

must be such that

$$\|(x(t), \lambda(t))\| \to 0, \qquad \text{asymptotically as } t \to \infty. \qquad ∎$$

REMARK 5. Since the exosystem dynamics $\dot{\lambda} = \tau(\lambda)$ is neutrally stable at $\lambda = 0$, it is immediate that $\lambda(t) \to 0$ as $t \to \infty$ implies $\lambda(0) = \lambda_0 = 0$. Hence, the zero-state detectability condition for plant (24) essentially requires the following: if $(x(t), \lambda(t))$ is any solution trajectory of (24) with small initial condition (x_0, λ_0), such that $y(t) = h(x(t), \lambda(t)) \equiv 0$, then $x(t) \to 0$ asymptotically as $t \to \infty$ and $\lambda_0 = 0$. ∎

Next, using Theorem 9, we establish the following result.

THEOREM 10. *Suppose that plant (24) satisfies the assumption*

$$f(0,\lambda) \equiv 0 \quad \text{and} \quad g(0,\lambda) \equiv 0, \qquad \text{for all } \lambda,$$

i.e., $x = 0$ is an equilibrium of the state dynamics for all values of the disturbance λ, and also that the plant output y is purely a function of x, i.e., it has the form

$$y = \gamma(x).$$

Suppose also that the exosystem dynamics $\dot{\lambda} = \tau(\lambda)$ is neutrally stable at $\lambda = 0$. Then there is no local asymptotic observer for plant (24).

PROOF. This is a simple consequence of Theorem 9. Basically, we show that plant (24) is not zero-state detectable. This is easily seen by taking $x(0) = x_0 = 0$ and $\lambda(0) = \lambda_0 \neq 0$. Then we have

$$y(t) = h(x(t), \lambda(t)) = \gamma(x(t)) \equiv 0.$$

Also, $x(t) \equiv 0$, since $x_0 = 0$. However, $\|\lambda(t)\|$ does not converge to 0 as $t \to \infty$ since $\lambda = 0$ is a neutrally stable equilibrium of the exosystem $\dot{\lambda} = \tau(\lambda)$. Thus, plant (24) is not zero-state detectable. By Theorem 9, it follows that there is no local asymptotic observer for plant (24). This completes the proof. ∎

COROLLARY 5. *Under the same hypotheses of Theorem 10, there is no local exponential observer for plant (24).* ∎

In sharp contrast to the case where $x = 0$ is an equilibrium for all values of the parameter λ, for the general case of problems, where we allow the equilibrium $x = 0$ to change with the parameter λ, we will show that there *generically* exist local exponential observers even when the output function y is purely a function of the state x, i.e., even when no information is available about the disturbance λ.

Next, we derive a characterization for the existence of local exponential observers of the form

$$\begin{aligned} \dot{z} &= \phi(z, \mu, y, u), \\ \dot{\mu} &= \psi(z, \mu, y, u) \end{aligned} \tag{25}$$

for plant (24), where z is defined in the neighborhood X of the origin of $\mathbb{R}^n$, and μ is defined in the neighborhood Θ of the origin of $\mathbb{R}^l$. Suppose that ϕ and ψ are locally C^1 mappings with $\phi(0,0,0,0) = 0$ and $\psi(0,0,0,0)$. If we define the estimation error e by

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \triangleq \begin{bmatrix} z \\ \mu \end{bmatrix} - \begin{bmatrix} x \\ \lambda \end{bmatrix},$$

then the error e satisfies the dynamics

$$\begin{aligned} \dot{e}_1 &= \phi(x + e_1, \lambda + e_2, y, u) - f(x, \lambda) - g(x, \lambda)u, \\ \dot{e}_2 &= \psi(x + e_1, \lambda + e_2, y, u) - \tau(\lambda). \end{aligned}$$

We consider the composite system

$$\begin{aligned} \dot{x} &= f(x, \lambda) + g(x, \lambda)u, \\ \dot{\lambda} &= \tau(\lambda), \\ \dot{e}_1 &= \phi(x + e_1, \lambda + e_2, h(x, \lambda), u) - f(x, \lambda) - g(x, \lambda)u, \\ \dot{e}_2 &= \psi(x + e_1, \lambda + e_2, h(x, \lambda), u) - \tau(\lambda). \end{aligned} \tag{26}$$

As in Section 4, we now suppose that the class $\mathcal{U}$ of admissible inputs consists of all inputs u of the form

$$u = r(\omega), \tag{27}$$

where $\omega \in \mathbb{R}^q$ satisfies the *input generator* dynamics given by

$$\dot{\omega} = s(\omega), \qquad \text{with } s(0) = 0. \tag{28}$$

We assume that the dynamics (28) is *neutrally stable* at $\omega = 0$, i.e., Lyapunov stable for both forward and backward time at $\omega = 0$. Thus, $\mathcal{U}$, the admissible class of input functions, includes constant signals and periodic signals with any desired period. We note that ω is available for measurement, being generated by the input generator, while the disturbance λ is usually not fully available for (direct) measurement.

Substituting (27) and (28) in the plant dynamics (24), we obtain the resulting plant as

$$\begin{aligned} \dot{x} &= f(x, \lambda) + g(x, \lambda)r(\omega), \\ \dot{\lambda} &= \tau(\lambda), \\ \dot{\omega} &= s(\omega), \\ y &= h(x, \lambda). \end{aligned} \tag{29}$$

A similar substitution in (26) leads to the composite system

$$\begin{aligned}
\dot{x} &= f(x,\lambda) + g(x,\lambda)r(\omega),\\
\dot{\lambda} &= \tau(\lambda),\\
\dot{\omega} &= s(\omega),\\
\dot{e}_1 &= \phi(x+e_1,\lambda+e_2,h(x,\lambda),r(\omega)) - f(x,\lambda) - g(x,\lambda)r(\omega),\\
\dot{e}_2 &= \psi(x+e_1,\lambda+e_2,h(x,\lambda),r(\omega)) - \tau(\lambda).
\end{aligned} \tag{30}$$

The following theorem holds for both classical and general cases of problems, and it can be proved similar to Theorem 3 as detailed in [14].

THEOREM 11. *Suppose that the plant dynamics in (29) is Lyapunov stable at* $(x,\lambda,\omega) = (0,0,0)$. *Then system (25) is a local exponential observer for plant (29) if, and only if,*

(a) *the submanifold defined via* $e = 0$ *is invariant under the flow of the composite system (30);*
(b) *the dynamics*

$$\begin{aligned}
\dot{e}_1 &= \phi(e_1,e_2,0,0),\\
\dot{e}_2 &= \psi(e_1,e_2,0,0)
\end{aligned}$$

is locally exponentially stable at $e = 0$. ∎

When we set $\omega = 0$ in (29), we get the unforced plant as

$$\begin{aligned}
\dot{x} &= f(x,\lambda),\\
\dot{\lambda} &= \tau(\lambda),\\
y &= h(x,\lambda).
\end{aligned} \tag{31}$$

As an application of Theorem 11, we establish the following result which states that when the plant dynamics in (29) is Lyapunov stable at $(x,\lambda,\omega) = (0,0,0)$, the existence of a local exponential observer for plant (29) in the presence of inputs implies and is implied by the existence of a local exponential observer for plant (29) in the absence of inputs.

THEOREM 12. *Suppose that the plant dynamics in (29) is Lyapunov stable at* $(x,\lambda,\omega) = (0,0,0)$. *If the system*

$$\begin{aligned}
\dot{z} &= \phi(z,\mu,y,u),\\
\dot{\mu} &= \psi(z,\mu,y,u)
\end{aligned}$$

is a local exponential observer for the full plant (29), then the system defined by

$$\begin{aligned}
\dot{z} &= \phi(z,\mu,y,0),\\
\dot{\mu} &= \psi(z,\mu,y,0)
\end{aligned}$$

is a local exponential observer for the unforced plant (31). Conversely, if the system

$$\begin{aligned}
\dot{z} &= \eta(z,\mu,y),\\
\dot{\mu} &= \sigma(z,\mu,y)
\end{aligned}$$

is a local exponential observer for the unforced plant (31), then the system defined by

$$\begin{bmatrix} \dot{z} \\ \dot{\mu} \end{bmatrix} = \begin{bmatrix} \phi(z,\mu,y,u) \\ \psi(z,\mu,y,u) \end{bmatrix} \triangleq \begin{bmatrix} \eta(z,\mu,y) \\ \sigma(z,\mu,y) \end{bmatrix} + \begin{bmatrix} g(z,y)u \\ 0 \end{bmatrix}$$

is a local exponential observer for the full plant (29).

PROOF. The calculations are similar to the proof of Theorem 4, and hence the proof is omitted. ∎

REMARK 6. For the class of nonlinear plants with exogenous disturbance and exogenous inputs considered in this paper, Theorem 12 asserts that under the Lyapunov stability assumption on the plant dynamics (which is quite natural to ask for as pointed out in Remark 2), the existence of a local exponential observer for the full plant (29) in presence of inputs implies, and is implied by the existence of local exponential observer for the full plant (29) in the absence of inputs. Theorem 12 also gives a simple construction procedure on constructing the local exponential observer for the full plant (29) from the local exponential observer for the unforced plant (31). This reduction procedure is extremely useful in applications, as it significantly simplifies the complexity of the exponential observer design problem for general nonlinear systems with exogenous disturbance, and exogenous inputs. In fact, we will show that under the Lyapunov stability assumption on the plant, a necessary and sufficient condition for the full plant (29) is that $(C^\star, A^\star)$ is detectable, where $(C^\star, A^\star)$ is the system linearization pair for the unforced plant (31). ∎

Let $(C^\star, A^\star)$ denote the linearization pair for the unforced plant (31), i.e.,

$$C^\star = \begin{bmatrix} C & Z \end{bmatrix} \qquad \text{and} \qquad A^\star = \begin{bmatrix} A & P \\ 0 & T \end{bmatrix},$$

where

$$C = \frac{\partial h}{\partial x}(0,0), \qquad Z = \frac{\partial h}{\partial \lambda}(0,0), \qquad A = \frac{\partial f}{\partial x}(0,0), \qquad P = \frac{\partial f}{\partial \lambda}(0,0), \qquad \text{and} \qquad T = \frac{\partial \tau}{\partial \lambda}(0).$$

In view of the reduction procedure outlined in Theorem 12, we first state some important results on the exponential observer design for the unforced plant (31). First, we state the following necessary condition for local exponential observers that can be easily established as in [16].

THEOREM 13. *If the unforced plant (31) has a local exponential observer, then the pair $(C^\star, A^\star)$ is detectable.* ∎

COROLLARY 6. *If the full plant (29) has a local exponential observer, then the pair $(C^\star, A^\star)$ is detectable.*

PROOF. The assertion follows immediately from Theorems 12 and 13. ∎

Using the necessary condition given in Theorem 13, we can establish the following result, which gives a simple necessary condition for the existence of local exponential observers for the unforced plant (31).

THEOREM 14. *If the unforced plant (31) has a local exponential observer, then*

(a) *the pair (C, A) is detectable;*

(b) *the pair $\left(\begin{bmatrix} Z \\ P \end{bmatrix}, T\right)$ is detectable.*

PROOF. Suppose that the unforced plant (31) has a local exponential observer. Then by Theorem 13, the pair $(C^\star, A^\star)$ is detectable. Note that by the PBH rank test [19, p. 286], a necessary and sufficient condition for $(C^\star, A^\star)$ to be detectable is that

$$\operatorname{rank} \begin{bmatrix} C^\star \\ \xi I - A^\star \end{bmatrix} = n + l, \tag{32}$$

for all complex numbers ξ in the closed right half plane (RHP).

Since

$$\begin{bmatrix} C^\star \\ \xi I - A^\star \end{bmatrix} = \begin{bmatrix} C & Z \\ \xi I - A & -P \\ 0 & \xi I - T \end{bmatrix},$$

it is immediate that (32) holds for all complex numbers ξ in the closed RHP only if

$$\operatorname{rank}\begin{bmatrix} C \\ \xi I - A \end{bmatrix} = n$$

for all complex numbers ξ in the closed RHP and

$$\operatorname{rank}\begin{bmatrix} Z \\ P \\ \xi I - T \end{bmatrix} = l.$$

for all complex numbers ξ in the closed RHP. This completes the proof. ∎

COROLLARY 7. *If the full plant (29) has a local exponential observer, then*

(a) *the pair (C, A) is detectable;*

(b) *the pair $\left(\begin{bmatrix} Z \\ P \end{bmatrix}, T\right)$ is detectable.*

PROOF. This is a simple consequence of Theorems 12 and 14. ∎

Next, we show that the necessary condition given in Theorem 12 is also sufficient for the existence of a local exponential observer for the unforced plant (31) when the unforced plant dynamics in (31) is Lyapunov stable at $(x, \lambda) = (0, 0)$.

THEOREM 15. *Suppose that the plant dynamics in (31) is Lyapunov stable at $(x, \lambda) = (0, 0)$, and suppose also that for some $(n + l) \times p$ matrix $K^\star$, the matrix $A^\star - K^\star C^\star$ is Hurwitz. Then the system defined by*

$$\begin{bmatrix} \dot{z} \\ \dot{\mu} \end{bmatrix} = \begin{bmatrix} f(z, \mu) \\ \tau(\mu) \end{bmatrix} + K^\star \left[y - h(z, \mu) \right] \tag{33}$$

is a local exponential observer for the unforced plant (31).

PROOF. It is easy to check that the candidate observer (33) satisfies Conditions (a) and (b) in Theorem 11. ∎

The next result makes use of the *reduction procedure* outlined in Theorem 12. When $(C^\star, A^\star)$ is detectable, by the reduction procedure outlined in Theorem 12, we can use the local exponential observer (33) constructed for the unforced plant (31) to construct a local exponential observer for the full plant (29).

THEOREM 16. *Suppose that the plant dynamics in (29) is Lyapunov stable at $(x, \lambda, \omega) = (0, 0, 0)$, and suppose also that for some $(n + l) \times p$ matrix $K^\star$, the matrix $A^\star - K^\star C^\star$ is Hurwitz. Then the system defined by*

$$\begin{bmatrix} \dot{z} \\ \dot{\mu} \end{bmatrix} = \begin{bmatrix} f(z, \mu) + g(z, \mu) u \\ \tau(\mu) \end{bmatrix} + K^\star \left[y - h(z, \mu) \right] \tag{34}$$

is a local exponential observer for the full plant (29).

PROOF. The assertion is a simple consequence of the reduction procedure outlined in Theorems 12 and 15. ∎

COROLLARY 8. *Suppose that the plant dynamics in (29) is Lyapunov stable at $(x, \lambda, \omega) = (0, 0, 0)$, and that the output function y is purely a function of x, i.e., it has the form $y = \gamma(x)$. Assume that the equilibrium $x = 0$ of the plant dynamics of the full plant (29) changes with the real parametric uncertainty λ. In this case, the system pair $(C^\star, A^\star)$ has the form*

$$C^\star = \begin{bmatrix} C & 0 \end{bmatrix} \qquad \textit{and} \qquad A^\star = \begin{bmatrix} A & P \\ 0 & T \end{bmatrix}.$$

If the pair $(C^\star, A^\star)$ is detectable, then the full plant (29) has a local exponential observer given by

$$\begin{bmatrix} \dot{z} \\ \dot{\mu} \end{bmatrix} = \begin{bmatrix} f(z,\mu) + g(z,\mu)u \\ \tau(\mu) \end{bmatrix} + K^\star \left[y - \gamma(z) \right], \tag{35}$$

where $K^\star$ is any matrix, such that $A^\star - K^\star C^\star$ is Hurwitz. ∎

REMARK 7. It is well known that the system linearization pair $(C^\star, A^\star)$ is *generically* observable [20]. Thus, from Corollary 8, we deduce that under the conditions:

(a) the plant dynamics in (29) is Lyapunov stable at $(x, \lambda, \omega) = (0, 0, 0)$;
(b) the equilibrium $x = 0$ of the plant dynamics in (29) changes with the disturbance λ;
(c) the output function y is purely a function of x, i.e., it has the form $y = \gamma(x)$,

there *generically* exist local exponential observers of form (35) for plant (29). ∎

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